

Classification of irreducible weight modules over W -algebra $W(2, 2)$ *

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Abstract. We show that the support of an irreducible weight module over the W -algebra $W(2, 2)$, which has an infinite dimensional weight space, coincides with the weight lattice and that all nontrivial weight spaces of such a module are infinite dimensional. As a corollary, we obtain that every irreducible weight module over the the W -algebra $W(2, 2)$, having a nontrivial finite dimensional weight space, is a Harish-Chandra module (and hence is either an irreducible highest or lowest weight module or an irreducible module of the intermediate series).

Key Words: The the W -algebra $W(2, 2)$, weight modules, support

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1. Introduction

The W -algebra $W(2, 2)$ was introduced in [ZD] for the study the classification of vertex operator algebras generated by weight 2 vectors.

Definition 1.1 *The W -algebra $\mathcal{L} = W(2, 2)$ is a Lie algebra over \mathbb{C} (the field of complex numbers) with the basis*

$$\{x_n, I(n), C, C_1 | n \in \mathbb{Z}\}$$

and the Lie bracket given by

$$[x_n, x_m] = (m - n)x_{n+m} + \delta_{n,-m} \frac{n^3 - n}{12} C, \quad (1.1)$$

$$[x_n, I(m)] = (m - n)I(n + m) + \delta_{n,-m} \frac{n^3 - n}{12} C_1, \quad (1.2)$$

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$$[I(n), I(m)] = 0, \quad (1.3)$$

$$[\mathcal{L}, C] = [\mathcal{L}, C_1] = 0. \quad (1.4)$$

The W -algebra $W(2, 2)$ can be realized from the semi-product of the Virasoro algebra Vir and the Vir-module $\mathcal{A}_{0,-1}$ of the intermediate series in [OR]. In fact, let $W = \mathbb{C}\{x_m \mid m \in \mathbb{Z}\}$ be the Witt algebra (non-central Virasoro algebra) and $V = \mathbb{C}\{I(m) \mid n \in \mathbb{Z}\}$ be a W -module with the action $x_m \cdot I(n) = (n - m)I(m + n)$, then $W(2, 2)$ is just the universal central extension of the Lie algebra $W \ltimes V$ (see [OR] and [GJP]). The W -algebra $W(2, 2)$ studied in [ZD] is the restriction for $C_1 = C$ of $W(2, 2)$ in our paper.

The W -algebra $W(2, 2)$ can be also realized from the so-called *loop-Virasoro algebra* (see [GLZ]). Let $\mathbb{C}[t, t^{-1}]$ be the Laurents polynomial ring over \mathbb{C} , then the loop-Virasoro algebra $\tilde{V}L$ is the universal central extension of the loop algebra $\text{Vir} \otimes \mathbb{C}[t^1, t^{-1}]$ and $W(2, 2) = \tilde{V}L/\mathbb{C}[t^2]$.

The W -algebra $W(2, 2)$ is an extension of the Virasoro algebra and is similar to the twisted Heisenberg-Virasoro algebra (see [ADKP]). However, unlike the case of the later, the action of $I(0)$ in $W(2, 2)$ is not simisimple, so its representation theory is very different from that of the twisted Heisenberg-Virasoro algebra in a fundamental way.

Next we recall the definitions of \mathbb{Z} -graded \mathcal{L} -modules. If \mathcal{L} -module $V = \bigoplus_{m \in \mathbb{Z}} V_m$ satisfies

$$\mathcal{L}_m \cdot V_n \subset V_{m+n}, \quad \forall m, n \in \mathbb{Z}, \quad (3.2)$$

then V is called a \mathbb{Z} -graded \mathcal{L} -module, and V_m is called a *homogeneous subspace of V with degree $m \in \mathbb{Z}$* .

A \mathbb{Z} -graded module V is called *quasi-finite* if all homogeneous subspaces are finite dimensional; a *uniformly bounded module* if there exists a number $n \in \mathbb{N}$ such that all dimensions of the homogeneous subspaces are $\leq n$; a *module of the intermediate series* if $n = 1$.

For any \mathcal{L} -module V and $\lambda \in \mathbb{C}$, set $V_\lambda := \{v \in V \mid x_0 v = \lambda v\}$, which we generally call the weight space of V corresponding the weight λ .

An \mathcal{L} -module V is called a weight module if V is the sum of all its weight spaces. For a weight module V we define

$$\text{Supp}(V) := \{\lambda \in \mathbb{C} \mid V_\lambda \neq 0\},$$

which is generally called the weight set (or the support) of V .

A nontrivial weight \mathcal{L} -module V is called a weight module of intermediate series if V is indecomposable and any weight spaces of V is one dimensional.

A weight \mathcal{L} -module V is called a highest (resp. lowest) weight module with highest weight (resp. highest weight) $\lambda \in \mathbb{C}$, if there exists a nonzero weight vector $v \in V_\lambda$ such that

- 1) V is generated by v as \mathcal{L} -module;
- 2) $\mathcal{L}_+ v = 0$ (resp. $\mathcal{L}_- v = 0$).

Remark. For a highest (lowest) vector v we always suppose that $I_0 v = c_0 v$ for some $c_0 \in \mathbb{C}$ although the action of I_0 is not semisimple.

Obviously, if M is an irreducible weight \mathbb{L} -module, then there exists $\lambda \in \mathbb{C}$ such that $\text{Supp}(M) \subset \lambda + \mathbb{Z}$. So M is a \mathbb{Z} -graded module.

If, in addition, all weight spaces M_λ of a weight \mathbb{L} -module M are finite dimensional, the module is called a *Harish-Chandra module*. Clearly a highest (lowest) weight module is a Harish-Chandra module.

Let $U := U(\mathcal{L})$ be the universal enveloping algebra of \mathcal{L} . For any $\lambda, c \in \mathbb{C}$, let $I(\lambda, c, c_0, c_1)$ be the left ideal of U generated by the elements

$$\{x_i, I(i) \mid i \in \mathbb{N}\} \bigcup \{x_0 - \lambda \cdot 1, C - c \cdot 1, I_0 - c_0 \cdot 1, C_1 - c_1 \cdot 1\}.$$

Then the Verma module with the highest weight λ over \mathcal{L} is defined as

$$M(\lambda, c, c_0, c_1) := U/I(\lambda, c, c_0, c_1).$$

It is clear that $M(\lambda, c, c_0, c_1)$ is a highest weight module over \mathcal{L} and contains a unique maximal submodule. Let $V(\lambda, c, c_0, c_1)$ be the unique irreducible quotient of $M(\lambda, c, c_0, c_1)$.

The following result was given in [ZD].

Theorem 1.2 [ZD] *The Verma module $M(\lambda, c, c_0, c_1)$ is irreducible if and only if $\frac{m^2-1}{12}c_1 + 2c_0 \neq 0$ for any nonzero integer m .*

The classification of Harish-Chandra modules over the W-algebra $W(2, 2)$ was given in [LLZ].

Theorem 1.3 [LLZ] *A Harish-Chandra module \mathcal{L} -module V is a highest weight module or lowest weight module or a module of the intermediate series.*

An irreducible weight module M is called a *pointed module* if there exists a weight $\lambda \in \mathbb{C}$ such that $\dim V_\lambda = 1$. Xu posted the following in [X]:

Problem 1.1 *Is any irreducible pointed module over the Virasoro algebra a Harish-chandra module?*

An irreducible weight module M is called a *mixed module* if there exist $\lambda \in \mathbb{C}$ and $i \in \mathbb{Z}$ such that $\dim V_\lambda = \infty$ and $\dim V_{\lambda+i} < \infty$. The following conjecture was posted in [M]:

Conjecture 1.2 *There are no irreducible mixed module over the Virasoro algebra.*

Mazorchuk and Zhao [MZ] gave the positive answers to the above question and conjecture to the Virasoro algebra, Shen and Su [SS] also gave a similar result for the twisted Heisenberg-Virasoro algebra.

In this paper, we also give the positive answers to the above question and conjecture for the W -algebra $W(2,2)$. Due to many differences between the $W(2,2)$ and the twisted Heisenberg-Virasoro algebra, some new methods are given in our paper.

Our main result is the following:

Theorem 1.4 *Let M be an irreducible weight L -module. Assume that there exists $\lambda \in \mathbb{C}$ such that $\dim M_\lambda = \infty$. Then $\text{Supp}(M) = \lambda + \mathbb{Z}$, and for every $k \in \mathbb{Z}$, we have $\dim M_{\lambda+k} = \infty$.*

The paper is organized as follow: Some lemma for the proof of Theorem 1.4 are given in Section 2. The Proof of the main Theorem is given in Section 3 where some corollaries from this theorem are also discussed.

2 Point modules over the W -algebra

We first recall a main result about the weight Virasoro-module in [MZ]:

Theorem 2.1 *Let V be an irreducible weight Virasoro-module. Assume that there exists $\lambda \in \mathbb{C}$, such that $\dim V_\lambda = \infty$. Then $\text{Supp}(V) = \lambda + \mathbb{Z}$, and for every $k \in \mathbb{Z}$, we have $\dim V_{\lambda+k} = \infty$.*

Lemma 2.2 *Assume that there exists $\mu \in \mathbb{C}$ and a non-zero element $v \in M_\mu$, such that*

$$I_1 v = L_1 v = L_{-1} I_2 v = L_2 v = 0 \quad \text{or} \quad I_{-1} v = L_{-1} v = L_1 I_{-2} v = L_{-2} v = 0.$$

Then M is a Harish-Chandra module.

Proof. Suppose that $I_1 v = L_1 v = L_2 v = 0$ for $v \in V_\mu$, it is clear that $L_{>0} v = 0$ and $I_m v = 0$ for $m \geq 3$. Moreover $L_{>0} I_2 v = 0$ and $I_m I_2 v = 0$ for $m \geq 3$ or $m = 1$.

But $L_{-1}I_2v = 0$, then $L_1L_{-1}I_2v = [L_1, L_{-1}]I_2v + L_{-1}L_1I_2v = -\frac{1}{2}L_0I_2v = 0$. So $I_2v = 0$ if $\mu \neq -2$. Then $\mathcal{L}_{>0}v = 0$. Hence v is a highest weight vector, and hence, M is a Harish-Chandra module.

If $\mu = -2$ and $w = I_2v \neq 0$, then $L_nw = L_nI_2v = [L_n, I_2]v + I_2L_nv = 0$ for any $n \in \mathbb{N}$. Moreover $I_1w = 0$ and $L_{-1}I_2w = [L_{-1}, I_2]I_2v + I_2L_{-1}I_2v = 0$. Then $I_2w = 0$ since $I_2w \notin V_0$. So $\mathcal{L}_{>0}w = 0$. Hence w is either a highest weight vector, and hence, M is a Harish-Chandra module. Similar for the lowest weight case. \square

Assume now that M is an irreducible weight \mathfrak{L} -module such that there exists $\lambda \in \mathbb{C}$ satisfying $\dim M_\lambda = \infty$.

Lemma 2.3 *There exists at most one $i \in \mathbb{Z}$ such that $\dim M_{\lambda+i} < \infty$.*

Proof. Assume that

$$\dim M_{\lambda+i} < \infty \quad \text{and} \quad \dim M_{\lambda+j} < \infty \quad \text{for some different } i, j \in \mathbb{Z}.$$

Without loss of generality, we may assume $i = 1$ and $j > 1$. Set

$$\begin{aligned} V := & \text{Ker}(I_1 : M_\lambda \rightarrow M_{\lambda+1}) \cap \text{Ker}(L_1, L_{-1}I_2 : M_\lambda \rightarrow M_{\lambda+1}) \cap \text{Ker}(I_j : M_\lambda \rightarrow M_{\lambda+j}) \\ & \cap \text{Ker}(L_j : M_\lambda \rightarrow M_{\lambda+j}), \end{aligned}$$

which is a subspace of M_λ . Since

$$\dim M_\lambda = \infty, \quad \dim M_{\lambda+1} < \infty \quad \text{and} \quad \dim M_{\lambda+j} < \infty,$$

we have, $\dim V = \infty$. Since

$$[L_1, L_k] = (k-1)L_{k+1} \neq 0 \quad \text{and} \quad [I_1, L_l] = (l-1)I_{l+1} \neq 0 \quad \text{for } k, l \in \mathbb{Z}, \quad k, l \geq 2,$$

we get

$$\begin{aligned} L_kV &= 0, \quad k = 1, j, j+1, j+2, \dots, \quad \text{and} \\ I_lV &= 0, \quad l = 1, j, j+1, j+2, \dots. \end{aligned} \tag{2.1}$$

If there would exist $0 \neq v \in V$ such that $L_2v = 0$, then $I_1v = L_1v = L_{-1}I_2v = L_2v = 0$ and M would be a Harish-Chandra module by Lemma 2.2. It is a contradiction. Hence $L_2v \neq 0$ for all $v \in V$. In particular,

$$\dim L_2V = \infty.$$

Since $\dim M_{\lambda+1} < \infty$, and the actions of I_{-1} and L_{-1} on L_2V map L_2V (which is an infinite dimensional subspace of $M_{\lambda+2}$) to $M_{\lambda+1}$ (which is finite dimensional), there exists

$0 \neq w \in L_2V$ such that $I_{-1}w = L_{-1}w = 0$. Let $w = L_2v$ for some $v \in V$. For all $k \geq j$, using (2.1), we have

$$L_k w = L_k L_2 v = L_2 L_k v + (2 - k) L_{k+2} v = 0 + 0 = 0.$$

Hence $L_k w = 0$ for all $k = 1, j, j + 1, j + 2, \dots$. Since

$$[L_{-1}, L_l] = (l + 1) L_{l-1} \neq 0 \quad \text{and} \quad [I_{-1}, I_l] = (l + 1) I_{l-1} \neq 0 \quad \text{for all } l > 1,$$

we get inductively $L_k w = I_k w = 0$ for all $k = 1, 2, \dots$. Hence M is a Harish-Chandra module by Lemma 2.2. A contradiction. The lemma follows. \square

Because of Lemma 2.3, we can now fix the following notation: M is an irreducible weight L -module, $\mu \in \mathbb{C}$ is such that $\dim M_\mu < \infty$ and $\dim M_{\mu+i} = \infty$ for every $i \in \mathbb{Z} \setminus \{0\}$.

Lemma 2.4 *Let $0 \neq v \in M_{\mu-1}$ and $\mu \neq -1$ such that $I_1 v = L_1 v = L_{-1} I_2 v = 0$. Then*

- (1) *There exists a nonzero $u \in M$ such that $L_1 v = I_m v = 0$ for all $m \geq 1$.*
- (2) *$I_m L_2 v = 0$ for all $m \geq 1$.*

Proof. Since $L_{-1} I_2 v = 0$, then $L_1 L_{-1} I_2 v = [L_1, L_{-1}] I_2 v + L_{-1} L_1 I_2 v = -\frac{1}{2} L_0 I_2 v = 0$. So $I_2 v = 0$ since $\mu \neq -1$. By $[L_1, I_k] = (k - 1) I_{k+1}$ we have $I_k v = 0$ for all $k \geq 2$. Moreover $I_m L_2 v = [I_m, L_2] v + L_2 I_m v = (2 - m) I_{m+2} v + L_2 I_m v = 0$. \square

Lemma 2.5 *Let $0 \neq w \in M_{\mu+1}$ and $\mu \neq 1$ such that $I_{-1} w = L_{-1} w = L_1 I_{-2} w = 0$. Then*

- (1) *$L_{-1} w = I_{-m} w = 0$ for all $m \geq 1$.*
- (2) *$I_{-m} L_{-2} w = 0$ for all $m \geq 1$.*

Proof. It is similar to that in Lemma 2.4. \square

3 Proof of Theorem 1.4

Proof of Theorem 1.4. Due to Lemma 2.3, we can suppose that $\dim M_\mu < +\infty$ and $\dim M_{\mu+i} = +\infty$ for all $i \in \mathbb{Z}, i \neq 0$.

Set

$$\begin{aligned} V := & \text{Ker}\{L_1 : M_{\mu-1} \rightarrow M_\mu\} \cap \text{Ker}\{I_1 : M_{\mu-1} \rightarrow M_\mu\} \\ & \cap \text{Ker}\{L_{-1} I_2 : M_{\mu-1} \rightarrow M_\mu\} \cap \text{Ker}\{L_{-1} L_2 : M_{\mu-1} \rightarrow M_\mu\} \subset M_{\mu-1}. \end{aligned}$$

For any $v \in V$, $L_1v = I_1v = L_{-1}I_2v = 0$

Since $\dim M_{\mu-1} = \infty$ and $\dim M_\mu < \infty$, we have $\dim V = \infty$. For any $v \in V$, consider the element L_2v . By Lemma 2.2, $L_2v = 0$ would imply that M is a Harish-Chandra module, a contradiction. Hence $L_2v \neq 0$, in particular, $\dim L_2V = \infty$.

Since the actions of I_{-1} , L_{-1} , L_1L_{-2} and L_1I_{-2} on L_2V map L_2V (which is an infinite dimensional subspace of $M_{\mu+1}$) to M_μ (which is finite dimensional), there exists $w = L_2v \in L_2V$ for some $v \in V$, such that $0 \neq w \in M_{\mu+1}$ and $I_{-1}w = L_{-1}w = L_1I_{-2}w = L_1L_{-2}w = 0$.

(1) If $\mu \neq \pm 1$, then

$$I_k w = 0, \quad k = 1, 2, \dots \quad (3.1)$$

from Lemma 2.4 and

$$I_{-k} w = 0, \quad k = 1, 2, \dots \quad (3.2)$$

from Lemma 2.5.

This means that I_k act trivially on M for all $k \in \mathbb{Z}$, and so M is simply an irreducible module over the Virasoro algebra. Thus, Theorem 1.3 follows from Theorem 2.1 in the case $\mu \neq \pm 1$.

(2) If $\mu = \pm 1$, we only show that $\mu = 1$ is not possible and for $\mu = -1$ the statement will follow by applying the canonical involution on \mathcal{L} .

In fact, if $\mu = 1$, then for $v \in V$, $L_1v = I_1v = L_{-1}I_2v = L_{-1}L_2v = L_0v = 0$. By Lemma 2.4, we have $I_kv = 0, k = 1, 2, \dots$.

For any $v \in V$, consider the element L_2v . By Lemma 2.2, $L_2v = 0$ would imply that M is a Harish-Chandra module, a contradiction. Hence $L_2v \neq 0$, in particular, $\dim L_2V = \infty$.

Since the actions of I_{-1} , L_{-1} and L_1I_{-2} on L_2V map L_2V (which is an infinite dimensional subspace of M_2) to M_1 (which is finite dimensional), there exists $w = L_2v \in L_2V$ for some $v \in V$, such that $w \neq 0$ and $I_{-1}w = L_{-1}w = L_1I_{-2}w = 0$. Moreover we have

$$I_k w = 0, \quad k = 1, 2, \dots \quad (3.3)$$

from Lemma 2.4. So

$$I_0 w = 0. \quad (3.4)$$

If $L_1w = L_1L_2v = 0$, then from $L_{-1}L_2v = 0$ we have $L_1L_{-1}L_2v = [L_1, L_{-1}]L_2v + L_{-1}L_1L_2v = 0$. So $L_0L_2v = 2L_2V = 0$ since $L_2v \in M_2$. Hence $L_2v = 0$ and then M is a highest weight module.

Then we can suppose that $L_1w \neq 0$ for any $w \in L_2V$.

For any $w \in L_2V$, consider the element $L_{-2}w$. If $L_{-2}w = 0$, then $L_{-k}w = I_{-k}w = 0, k = 1, 2, \dots$. Then M is a Harish-Chandra module. Hence $L_{-2}L_2V \neq 0$, in particular,

$\dim L_{-2}L_2V = \infty$. Let $W = L_2V$, then L_1 maps $L_{-2}W$ to M_1 has infinite dimensional Kernel K . Let $0 \neq L_{-2}w \in K$, then $L_1L_{-2}w = 0$. But $L_1L_{-2} = [L_1, L_{-2}] + L_{-2}L_1$ and $[L_1, L_{-2}]w = (-3)L_{-1}w = 0$, hence $L_{-2}L_1w = 0$. Setting $u = L_1w \neq 0$, we have $L_{-2}u = 0$, $I_{-1}u = I_{-1}L_1w = [I_{-1}, L_1]w + L_1I_{-1}w = 0$. Moreover by induction we have $I_{-m}u = 0$ for all $m \geq 3$.

$I_mu = [I_m, L_1]w + L_1I_mu = (1-m)I_{m+1}w + L_1I_mu = 0$ for all $m \geq 0$ by (3.3) and (3.4). So $I_2u = \frac{1}{6}[L_{-2}, I_4]u = 0$. Then $I_ku = 0$ for all $k \in \mathbb{Z}$.

By $L_{-2}u = 0$, we have $I_2L_{-2}u = 0$. Therefore $c_1 = 0$.

This means that $I_k, k \in \mathbb{Z}, C_1$ act trivially on the irreducible M for all $k \in \mathbb{Z}$, and so M is simply a module over the Virasoro algebra. Thus, Theorem 1.3 follows from Theorem 2.1. \square

Theorem 1.4 also implies the following classification of all irreducible weight L -modules which admit a nontrivial finite dimensional weight space:

Corollary 3.1 *Let M be an irreducible weight L -module. Assume that there exists $\lambda \in \mathbb{C}$ such that $0 < \dim M_\lambda < \infty$. Then M is a Harish-Chandra module. Consequently, M is either an irreducible highest or lowest weight module or an irreducible module from the intermediate series.*

Proof. Assume that M is not a Harish-Chandra module. Then there should exists $i \in \mathbb{Z}$ such that $\dim M_{\lambda+i} = \infty$. In this case, Theorem 1.3 implies $\dim M_\lambda = \infty$, a contradiction. Hence M is a Harish-Chandra module, and the rest of the statement follows from Theorem 1.3. \square

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